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THE CALENDAR

BY

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1. Our Calendar is the result of ages of human thought and experience. The ancient Egyptians, who built the pyramids about 5000 years ago, were busy solving the Calendar Problem, and it cannot be said to have been completely solved even now. I shall confine myself to the three Calendars in use at present in Northern India, viz. the Christian, the Vikrama or the Hindu, and the Hijri Calendars and shall try to explain the main principles on which they are based.

2. It is a matter of common knowledge now that the revolution of the earth round the sun is responsible for the changes in seasons. The ancients did not know this. When they took to agriculture for their living, they wanted to know when they were to sow and when the rains were to be expected. The moon with her changing phases came to their help. They found out that between one rainy season and the next there was a certain number of new moons. It was soon noticed that this information was not sufficient. Having no observatories or scientific instruments, it took them very long to discover that the seasons came in a cycle of about 360 days, or that the sun rose at the same place after an interval of about 360 days. Thus the track of the sun in the heavens was mapped out in 360 steps each corresponding to a day and a night. Incidentally it may be remarked that our present method of dividing the circumference of a circle into 360 equal "degrees," is a natural development of this simple procedure. Later it was found that this estimate was not correct. The addition of five feast days to the twelve Egyptian months of 30 days each seems to point to the recognition of this error. This new estimate is nearly correct. The Hindus estimated the length of the year at 365 days 6 hours and 12 minutes nearly. Considering the difficulties under which they laboured and an almost entire lack of scientific instruments available to modern investigators, this was a great achievement indeeed. We know now that this is slightly in excess, as it takes the earth 365 days 5 hours 48 minutes and 46 seconds, to go round the sun and come back to its original position in its orbit. This period of time is called the "Solar Year." Again it was found that between one new moon and the next one the interval was about 29½ days. This period came to be called a "month" on account of its associ-

* Based on a paper read at a meeting of the Study Circle, Hoshiarpur.

ation with the moon. In fact a lunar month is 29 days 12 hours 44 minutes and 3 seconds approximately. Gradually, therefore, the word "month" came to be used for a period of about 30 days.

3. It is well-known that the length of the day the period of time between one sun-rise and the next is not always the same except at places on the equator. We divide a day into 24 hours, each hour into 60 minutes and each minute into 60 seconds. The Hindus have a better and more uniform way of dividing the day into 60 gharis, each ghari into 60 pals and each pal into 60 vipals. A ghari is 2/5 of an hour, a pal 2/5 of a minute and a vipal 2/5 of a second. It will be interesting to note that the ratio 2/5 seems to be of some significance in dealing with Indian measures, thus one Kachcha seer is 2/5 of 1 Pakka seer. Thus understood a ghari may be called a Kachcha hour, a pal a Kachcha minute and a vipal a Kachcha second.

4. The Christian Calendar.

The present Christian year is "solar." It was once "Luni-Solar," i.e. it followed the moon so far as the months were concerned and the sun for the seasons. At intervals an extra month was intercalated in order to keep the calendar in pace with the seasons. The present Hindu "Luni-Solar" Calendar does the same. Julius Ceasar, with the advice and assistance of Sosigenes, fixed the mean length of the year at 365¼ days and laid down that every fourth year should have 366 days others having 365. This made the mean year too long by 11 minutes 14 seconds, the error accummulating to nearly 3 days in the course of 400 years. Pope Gregory XIII therefore directed in March, 1582 that 10 days be suppressed in the Calendar and ordered that February should have only 28 days in all centenary years excepting those that are multiples of 400. Thus the year 1900 was not a leap year, while the year 2000 will be. Even this arrangement leaves the average year too long by 26 seconds. In 3323 years the excess will amount to one day. It has, consequently, been proposed to correct the "Gregorian Rule" by making the year 4000 an ordinary instead of a leap year. It would have been very much better if Pope Gregory XIII had ordered that February should have 29 days in all years which are multiples of 4 but not of 128. We would thus have had 31 leap years, in 128 years. This would have left the average year in defect only by 1 second. It is a pity that he did not so ordain.

The Christian Calendar has 12 months of unequal lengths; seven having 31 and four 30 days each, and February having 28 in the ordinary and 29 in the leap year. It is believed that the months were once alternately of 31 and 30 days each except February which had 29 in the ordinary and 30 days in the leap year. It may be of interest to mention that the year originally began with March; September, October, November and December were so named because they were respectively the 7th, 8th, 9th and the 10th months of the year. July was called Quintilis and August Sextilis. Quintilis was re-named July in order to please Julius Caesar while Sextilis was rechristened August to gratify

Augustus. It is further believed that August was allotted 31 days simply because Julius Caesar's July had 31. Augustus could not bear that his month should have fewer days than Julius's. Several attempts were made to change the names of the other months to humour kings and emperors, but evidently they failed. When August was given 31 days, poor February (the last month of the year) had another day knocked off. In recent years attempts have been made to revise the Calendar to suit the needs of the commercial and wage-earning communities. It has, for instance, been proposed to have 13 months of 28 days each, with a general holiday on the New Year's Day which is to have no date or dayname. The Leap Year's Day is to be a holiday in the same manner. Every month is to begin with a Sunday. National Festivals are to be inserted on Saturdays or Mondays once for all, so that the people may combine the holidays to be better able to enjoy themselves. One reason why these reforms have not been accepted is that the superstitious West does not like the idea of having 13 months on account of the ominous associations of that number.

To sum up, the main points about the present Christian Year are:—

(1) It is "Solar". (2) The lengths of the months are purely arbitrary and have no scientific basis. (3) The quarters are not of the same length. (4) The day is from mid-night to mid-night.

It may be remarked that the Gregorian Reform was not immediately accepted. Russia was the last to do so under the Soviet Government. It appears that "inertia" is as much applicable to the laws of the mind as to those of the physical world.

5. The Vikrama Calendar.

It is of two different types :— (1) Solar (2) Luni-Solar. In the Luni-Solar Calendar we have "tithis" (dates of the moon), as the basis of reckoning. Every three years a month is intercalated in the lunar year so as to make it keep pace with the solar year.

The Solar year has, as already stated, 365 days, six hours, 12 minutes and 36·56 seconds, according to the present Surya Siddhanta. It is divided into 12 months of different durations. The time taken by the earth in revolving round the sun through an angle of 30° is termed a "month" in the solar year of the Hindus. If the path of the earth were a circle with the sun at the centre, the months would all be of equal length ; but as the orbit of the earth is an ellipse and the sun is in one of the foci, the earth takes different periods of time in revolving through angles of 30° round the sun at different times of the year. The months have different lengths on this account. Thus according to the present Surya Siddhanta the length of the month of *Asaḍh* is 31 days, 15 hours 28 minutes and 24 seconds approximately, whereas the length of the month of *Phalguṇa* is only 29 days, 19 hours, 41 minutes 12 seconds. The exact time at which a new month begins is termed as "Samkranti". The day is

reckoned from sun-rise to sun-rise, the point at the equator on the longitude of the place being taken as the basis for the reckoning. The ancient Indian almanacs took a place on the equator at the longitude of Ujjain as their basis and all times were calculated from the sun-rise at the place. Ujjain may be called the Greenwich of India. As a convention it was enjoined that the first of the month should be considered to have fallen on the day during the course of which the "Samkranti" occurs. Thus if the Samkranti were to occur even a few minutes before equatorial sunrise tomorrow, today will be regarded as the first of the month. This convention is very sound and leaves no scope for leap years. It will thus be clear that Hindu months can have days from 29 to 32. In general the summer months have more days than the winter months.

The Hindu Calendar needs reform in two directions. First, it should begin with the "vernal equinox" as it was actually intended to. Secondly, the length of the year should be corrected. The length of the year as stated in the Surya Siddhanta is in excess of the correct length by nearly 24 minutes. We are thus moving ahead at the rate of one day in every 60 years. If the necessary reform is not affected the first of *Vaishakh* may come in winter one day. It is time that we rectified the error. Opposition there is bound to be, but the earlier an error is corrected the better. In all other respects the Hindu Calendar is more scientific and more in conformity with nature than the Christian Calendar in use at present; because the months are not arbitrary in length, and because the day begins with the most charming natural event—the sun-rise, and not at mid-night when the world is plunged in darkness.

6. The Hijri Calendar.

The era commences with the flight of the prophet from Mecca to Medina, which took place on Friday, the 19th July, 622 A. D. according to the reformed calendar. The Hijri year is purely lunar. It has 12 months of 29 days, 12 hours, 44 minutes and 3 seconds each. In practice the months have 30 and 29 days alternately. This leaves the year too short by 8 hours, 48 minutes and 36 seconds. In 30 years the difference amounts to about 11 days. Umar Khayyam, the poet and mathematician, therefore hit upon a cycle of 30 years in which 19 years have 354 days each and 11 years have 355 days each. The intercalary years are the 2nd, 5th, 7th, 10th, 13th, 16th, 18th, 21st, 24th, 26th, and 29th. This arrangement is very satisfactory in so far as it leaves the average Hijri year too short only by 36 seconds. However, Umar Khayyam's plan seems never to have been adopted in practice and the appearance of the moon has always been a natural event of importance in the Hijri year. The month begins with the appearance of the moon and the first of the month falls on the day following the appearance of the moon. In the Hijri Calendar the day is calculated from sunset to sunset. It will, thus, be readily seen that there is an element of great uncertainty in the Hijri Calendar and even the almanacs for any year are not wholly trustworthy. I have made an attempt*

* See my paper on the "Indian Calendar" in *Popular Astronomy*, 1934, Vol. 42 pp. 82-84.

towards removing this uncertainty by taking into consideration the 36 seconds in a year left out by Umar Khayyam. It is satisfying to note that my calendar for Hijri years gives results which are more in agreement with the observations and the prevailing conventions than those given by the almanac makers. It will not be out of place to remark that the Hijri months are not arbitrary; they are wholly based on natural events. Since the Hijri year is purely lunar, its months give no idea of the seasons. In about 33 years' time the Hijri months make a round through the solar year.

GLEANINGS

HEIGHTS AND DISTANCES

He further exhibited with great pride a mass of figures which he had worked out upon the mirror of Mr. Purvale's chiffonier with a stick of shaving soap. "So that," he went on, "by further measuring the exact deadline between where the bullets entered the room, to where they struck the wall, and then right-angling the tangent we have got the exact point of line of fire as so many feet higher, which'd make it some window on the other side of the street, and so many feet to the right or left of that, as it may be. You can see that clearly, can't you?" he demanded. "I tell you, my lad, you learn something when you've got to read for the Army. Even whether you pass your exams or not," he amended somewhat lamely.

From: J. G. BRANDON, IN *The Glass Dagger.*
Per A. A. Krishnaswami Ayyangar, *Mysore.*

The property of attraction is inherent in the earth. By this property, the Earth attracts any unsupported heavy thing towards it. That thing appears to be falling, but it is in a state of being drawn to the Earth. The etherial expanse being equally outspread all round, where can the Earth fall? It is manifest from this that neither can the Earth by any means fall downwards nor the men situated at the distance of a fourth part of a circumference from us or in the opposite hemisphere.

From BHASKARA: *Goladhyaya.*

As the one–hundred part of the circumference of a circle is scarcely different from a plane and the Earth is an excessively large body and a man exceedingly small in comparison, consequently the whole visible portion of the Earth appears to man on its surface to be perfectly plane.

From BHASKARA: *Goladhyaya.*

SOME THEOREMS CONCERNING QUINTICS INSOLUBLE BY RADICALS

BY

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In what follows all the latin letters denote integers.

The equation

$$x^5 + cx + d = 0 \qquad \dots \quad (1)$$

if irreducible, is insoluble by radicals if

$$(z - c)^4(z^2 - 6cz + 25c^2) = 5^5d^4z \qquad \dots \quad (2)$$

has an integral root. (Cajori's, *Theory of Equations*). Recently we* have constructed similar tests, for the equations

$$x^5 + ax^2 + bx + c = 0 \qquad \dots \quad (3)$$

and

$$x^5 + ax^3 + bx^2 + cx + d = 0 \qquad \dots \quad (4)$$

The following theorems concerning quintics insoluble by radicals are easy to prove, but so far as we know, they have never been stated before.

THEOREM 1. $x^5 + ax^3 + bx^2 + cx + d = 0$
(if irreducible) is insoluble by radicals if

$$a \equiv b \equiv 0 \pmod 2$$

and

$$c \equiv d \equiv 1 \pmod 2.$$

This is easily deduced from the condition analogous to (2) for (4). As a special case it follows that $x^5 + cx + d = 0$ is insoluble by radicals is $c \equiv d \equiv 1 \pmod 2$.

THEOREM 2. $x^5 + cx + d = 0$ is insoluble by radicals if c is odd and not divisible by any prime $\equiv 3 \pmod 4$.

Proof: From equation (2) we have that $z \mid 25c^6$.

Hence z must be odd and $\equiv 1 \pmod 4$; so that the left side of (2) is $\equiv 0$ (mod. 16) and d must be even.

* S. Chowla, in an unpublished note, has obtained the condition analogous to (2) for (3), and Y. Bhalotra for (4) in *The Mathematics Student*, Dec. 41.

It follows that

$$z^2 - 6cz + 25c^2 \equiv 0 \pmod{16}$$

i.e. $$(z-3c)^2 + 16c^2 \equiv 0 \pmod{16}.$$

Hence, $$z \equiv 3c \pmod 4$$

$$\equiv 3 \pmod 4,$$

contrary to $z \equiv 1 \pmod 4$, proved earlier.

THEOREM 3. $x^5 + cx + d = 0$ is insoluble by radicals if (*i*) c is a prime $\not\equiv 1 \pmod 5$ and (*ii*) $(d, c) = 1$.

Proof. Let us write $c^{\alpha} \| z$ to denote $c^{\alpha} | z$ but $c^{\alpha+1}$ is not a divisible by z.

Then it is clear from the equation (2) that if $\alpha > 0$, then $\alpha \geq 6$.

Now $z | 25c^6$.

$\therefore$ z can have only the values
$z = 1, 5, 5^2, c^6 \cdot 5, c^6 \cdot 5^2$.

Let us suppose that $z = 5c^6$. Then (2) becomes

$$(5c^5 - 1)^4 \, (5c^{10} - 6 \cdot c + 5) = 5^5 \, d^4$$

Now $5c^5 - 1 \not\equiv 0 \pmod 5$

$$5c^{10} - 6 \cdot c^5 + 5 \not\equiv 0 \pmod 5 \text{ if } c \neq 5.$$

If $c = 5$, then

$$5c^{10} - 6c^5 + 5 = 5\,(5^{10} - 6 \cdot 5^4 + 1)$$

and $$5^{10} - 6 \cdot 5^4 + 1 \not\equiv 0 \pmod 5$$

In either case $5c^{10} - 6c^5 + 5 \not\equiv 0 \pmod{5^5}$

Hence the left side $\not\equiv 0 \pmod{5^5}$ while
right side $\equiv 0 \pmod{5^5}$.

Hence $z = 5c^6$ is not permissible.

If $z = 5^2 \cdot c^6$, (2) becomes

$$(5^2 \, c^5 - 1)^4 \, (5^2 \cdot c^{10} - 6 \cdot c^5 + 1) = 5^3 \, d^4.$$

Now $5^2 c^5 - 1 \not\equiv 0 \pmod{5}$

and
$$5^2 c^{10} - 6c^5 + 1 \equiv 1 - 6c^5 \pmod{5}$$
$$\equiv 1 - 6c \pmod{5}$$
$$\not\equiv 0 \pmod{5}.$$

Hence the left side cannot contain the factor 5 which exists on the right side.

Similarly we can show that other values of z do not hold.

Hence the theorem enunciated.

GLEANINGS

If a piece of cloth be cut in a circular form with a diameter equal to half the circumference of the sphere, then half the sphere will be entirely covered by that circular cloth and there will be some cloth to spare. As the area of this piece of cloth is to be found nearly 2½ times the area of a great circle of the sphere, and the area of the piece of cloth covering the other half of the sphere is also the same, the area of the whole sphere cannot be more than 5 times the area of the great circle of the sphere. How then has he (Lalla) multiplied the area of the great circle of the sphere by the circumference to get the superficial contents of the sphere? As the area of the great circle of the sphere multiplied by the circumference is without reason, the rule given by Lalla is wrong and consequently surface area of the Earth is wrong.

From BHASKARA : *Goladhyaya.*

In answering this question Professor C. E. M. Joad says that because of science we have been brought to think that a thing to be real must be comprehended by our sense organs like touch, sight, etc. This limitation of science is unwarranted. There are other orders of Reality, which though not material, are none the less real. To these belong mathematics, music and chess. Continues the Professor, " It may be that the soul has inhabited such an order of Reality before it was incarnated in the body and brings with it to this world a memory of the harmonies of sound and pattern and combinations of number which exist in those orders of Reality: the chess genius has perhaps never perfected himself in the pursuits of or obtained familiarity with the things of the terrestrial sensory orders of Reality: hence he is no good at business or the professions.

T. A. KRISHNAMACHARIAR IN *The Hindu.*

ON SYMMETRIC POLYNOMIAL FUNCTIONS OF ZEROS OF POLYNOMIALS

BY

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The object of this note is to give a proof of the well known fundamental theorem on polynomial symmetric functions of zeros of polynomials. The statement of the theorem as it is given in this note is slightly sharper than what one meets with in many elementary textbooks and the proof avoids any reference to the derivative of a polynomial. It is possible that the undergraduate students who first calculate, in terms of the coefficients of a polynomial $P(z)$, the value of a number of simpler symmetric functions of the zeros of $P(z)$ will find the proof given here to be more 'natural'.

Let

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

and

$$R = \sum \alpha_{r_1}^{\varepsilon_1} \alpha_{r_2}^{\varepsilon_2} \cdots \alpha_{r_t}^{\varepsilon_t}$$

where $r_1, r_2, \cdots r_t$ are distinct (positive) integers, $\alpha_1, \alpha_2, \cdots \alpha_n$ are the n zeros of $P(z)$, $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t$ are non-negative integers, $1 \leqslant t \leqslant n$, $1 \leqslant r_k \leqslant n$ $(k=1, 2, \cdots t)$, and the summation is over all such permutations of the suffixes as give rise to different terms. For example, if $n=3$ then we can take R to be such functions as

$$\alpha_1\alpha_2\alpha_3,$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3,$$

$$\alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2.$$

and so on. Such symmetric functions of $\alpha_1, \alpha_2, \alpha_n$ are called here typical symmetric functions; plainly, any symmetric polynomial in $\alpha_1, \alpha_2, \cdots \alpha_n$ is a linear function of typical symmetric functions. We represent the typical symmetric function R by $(\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t)$, and suppose, without loss of generality, that $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_t$. The degree of R is defined to be $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t$. We specify below a scheme for enumerating (i.e., ranking) typical symmetric functions. If R_1 and R_2 are two typical symmetric functions of degrees d_1 and d_2

respectively and $d_1 \neq d_2$ then R_1 is of a higher or lower rank than R_2 according as $d_1 > d_2$ or $d_1 < d_2$; if $d_1 = d_2$ but R_1 and R_2 are distinct,

$$R_1 = \{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t\},\ R_2 = \{\eta_1, \eta_2, \cdots \eta_s\},$$

$(\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_t,\ \eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_s)$ then R_1 is of a higher or lower rank than R_2 according as the first nonvanishing number of the set $\varepsilon_1 - \eta_1, \varepsilon_2 - \eta_2, \cdots$ is positive or negative. For example, if $n = 3$ then the enumeration of typical symmetric functions is as follows :—

$$T_1 = \sum \alpha_r = \alpha_1 + \alpha_2 + \alpha_3,$$
$$T_2 = \sum \alpha_{r_1}\alpha_{r_2} = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3,$$
$$T_3 = \sum \alpha_r^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$
$$T_4 = \sum \alpha_{r_1}\alpha_{r_2}\alpha_{r_3} = \alpha_1\alpha_2\alpha_3$$
$$T_5 = \sum \alpha_{r_1}^2\alpha_{r_2} = \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2.$$
$$T_6 = \sum \alpha_r^3 = \alpha_1^3 + \alpha_2^3 + \alpha_3^3.$$
$$T_7 = \sum \alpha_{r_1}^2\,\alpha_{r_2}\,\alpha_{r_3} = \alpha_1^2\,\alpha_2\,\alpha_3 + \alpha_2^2\,\alpha_1\,\alpha_3 + \alpha_3^2\,\alpha_1\,\alpha_2, \text{ and so on.}$$

THEOREM :—*The value of any typical symmetric function R is equal to $Q_R(a_1, a_2, \cdots a_n)$ where $Q_R(a_1, a_2, \cdots a_n)$ is a polynomial in $a_1, a_2, \cdots a_n$ and has integer coefficients.*

Remark. It is obvious that the same conclusion holds for any symmetric polynomial function provided that its coefficients are integers.

Plainly the theorem is true for T_1, and T_2 and, more generally, for such T's as are of the form $\{1, 1, \cdots\}$. We prove the theorem by induction. Suppose now that the theorem is true for $T_1, T_2, \cdots T_m$, and let

$$T_{m+1} = \{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t\} \qquad (\varepsilon_r \geqq 1,\ r = 1, 2, \cdots t).$$

It is easy to verify that

$$T_{m+1} = \{\varepsilon_1 - 1, \varepsilon_2 - 1, \cdots \varepsilon_t - 1\} \times \{1, 1, \cdots 1\} - \sum_{r=1}^{m} C_r\, T_r$$

Where the number of 1's in the second factor of the product is equal to t, and $c_1, c_2, \cdots c_m$ are (non-negative) integers which depend on n and m. From the inductive hypothesis we see that the right hand side of the above equation is a polynomial in $a_1, a_2, \cdots a_n$ and that its coefficients are integers. This completes the proof of the theorem.

A PORISM ON ELEVEN SPHERES

BY

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1. Let (R) be the orthogonal sphere of the four spheres (A), (B), (C), (D), centers A, B, C, D, and (L) another given sphere. The sphere (A′) is constructed orthogonal to (A) and passing through the circle of intersection, real or imaginary, of the sphere (L) and the plane BCD; let (B′), (C′), (D′) be the analogous spheres relative to the spheres (B), (C), (D).

Porism. The radical center M of the four spheres (A′), (B′), (C′), (D′) is the harmonic pole, for the tetrahedron (T) = ABCD, of the radical plane μ of the spheres (L) and (R).

Let *the radical plane μ meet the edges* DA, DB, DC, BC, CA, AB of the tetrahedron (T) in the points U, V, W, X, Y, Z, respectively, and let U′, . . . , Z′ be the harmonic conjugates of U, . . . , Z for the respective pairs of vertices of (T).

The lines U′X′, V′Y′, W′Z′ have a point, say M, in common which is the harmonic pole of the plane μ = UVWXYZ for the tetrahedron (T)[1]. Remains to be shown that M coincides with the radical center of the four spheres (A′), (B′), (C′), (D′).

The planes BCD, ACD are respectively the radical planes of the pairs of spheres (A′) and (L), (B′) and (L), hence the radical plane of (A′) and (B′) passes through the line CDWW′. Again, the planes μ, BCD are the radical planes of the pairs of spheres (L) and (R), (L) and (A′), hence the radical plane of the spheres (R) and (A′) passes through the line XVW = (μ, BCD). Now both (A′) and (R) are orthogonal to the sphere (A), hence the center A of (A) lies in the radical plane of (A′) and (R), and therefore the radical plane of (A′) and (R) coincides with the plane A-XVW. Similarly the radical plane of (B′) and (R) coincides with the plane B-YUW. Consequently the radical plane λ of (A′) and (B′) passes through the line of intersection of the planes A-XVW, B-YUW.

It follows immediately that the point W belongs to the radical plane λ of (A′) and (B′), as has been noticed already. Moreover, the line AV of the plane A-XVW and the line BU of the plane B-YUW

both lie in the plane ABD, hence they have a point, say K, in common, and K belongs to the plane λ. Now the line DK meets AB in the point Z', and both K and D belong to λ, hence Z' also lies in λ. Thus the radical plane λ of the spheres (A'), (B') passes through the points Z' and W', hence λ passes through the point M.

Similarly for the radical plane of any other two of the four spheres (A'), (B'), (C'), (D'). Consequently the radical center of these four spheres coincides with M, which proves the proposition.

The above porism gives rise to a large number of propositions. Some examples will be given in what follows.

2. We may take for (L) the circumsphere (O) of the tetrahedron (T) = ABCD, and for (A'), (B'), (C'), (D') the spheres having for great circles the circumcircles of the triangles BCD, CDA, DAB, ABC, respectively. The radical center of these four spheres is the isogonal conjugate, with respect to (T), of the circumcenter O of (T)[2].

Then (A), (B), (C), (D) are the spheres having A, B, C, D for centers and orthogonal respectively to the spheres (A'), (B'), (C'), (D'). The radical plane of the orthogonal sphere (R) of (A), (B), (C), (D) with the circumsphere (O) = (L) of (T) is the newtonian plane of the four spheres.[3] We thus have the proposition:

If with the vertices of a tetrahedron (T) *as centers spheres* (A), (B), (C), (D) *are drawn respectively orthogonal to the spheres having the circumcircles of the faces of* (T) *for diametral circles, the newtonian plane of the spheres* (A), (B), (C), (D) *has for harmonic pole with respect to* (T) *the isogonal conjugate, for* (T), *of the circumcenter of* (T).

3. a. Suppose we take for (A'), (B'), (C'), (D') the spheres passing through a given point M and through the circles of intersection of the faces of a tetrahedron (T) = ABCD with the given sphere (L). The point M is the radical center of the four spheres passing through M, and the harmonic plane of M for (T) will be the radical plane of (L) and the orthogonal sphere (R) of the four spheres (A), (B), (C), (D) having A, B, C, D for centers and orthogonal respectively to (A'), (B'), (C'), (D').

b. Suppose further that M coincides with the centroid G of (T), and that (L) is concentric with the circumsphere (O) of T.

The radical plane of (R) and (L) is the harmonic plane of G for (T), i. e., the plane at infinity, consequently the spheres (R) and (L) are concentric. Thus the orthogonal sphere (R of (A), (B), (C), (D)

is concentric with the circumsphere (O) of (T), i. e., the center of (R) is the circumcenter O of (T).

Let a, b, c, d, r, be the radii of the spheres (A), (B), (C), (D) (R). We have:

$$a^2+r^2=OA^2, \cdots, \cdots, d^2+r^2=OD^2;$$

but $OA=OB=OC=OD$, hence $a=b=c=d$.

Thus: *The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.*

c. If we assume that (L) coincides with the circumsphere (O) of (T), the above proposition may be stated as follows: The powers of the vertices of a tetrahedron with respect to the spheres determined by the centroid and the remaining three vertices, are equal[4].

4. May we remind the reader that the six points of intersection of the six edges of a tetrahedron (T) with the external bisecting planes of the respectively opposite dihedral angles of (T) lie in a plane.[5] We shall denote that plane by γ. The harmonic pole of γ for (T) is the in-center I of (T).[6]

Now suppose we take for (A), (B), (C), (D) the spheres having for centers the vertices of a tetrahedron (T) = ABCD and passing respectively through the points of contact A′, B′, C′, D′ of the opposite face of (T) with the inscribed sphere (I) of (T).

Suppose further that we take for (L) the sphere (I). The spheres (A′), (B′), (C′), (D′) thus reduce to the points A′, B′, C′, D′. The orthogonal sphere of the four point spheres thus coincides with (I), hence their radical center coincides with the center I of (I). Consequently the harmonic plane γ of I for (T) is the radical plane of the sphere (I) = (L) with the orthogonal sphere (R) of the four spheres (A), (B), (C), (D); hence γ is perpendicular to the line joining the centers R, I of the spheres (R), (I). Thus:

The line joining the incenter of a tetrahedron (T) to the radical center of the four spheres having for centers the vertices of (T) and passing through the points of contact of the respectively opposite face of (T) with its inscribed sphere is perpendicular to the plane γ of (T).

Analogous propositions may be formulated for the other quadritangent spheres of (T).

5. Suppose we take four spheres (A), (B), (C), (D) with radii equal to zero. Their orthogonal sphere (R) coincides with the circumsphere (O) of the tetrahedron (T) = ABCD formed by their centers, and our porism becomes:

The radical center of the four spheres determined by the vertices of a tetrahedron (*T*) *and the circles of intersection of the respectively opposite faces of* (*T*) *with a given sphere* (*L*), *is the harmonic pole, for* (*T*), *of the radical plane of* (*L*) *and the circumsphere* (*O*) *of* (*T*).

This proposition is due to the Belgian mathematician J. Neuberg (1840 - 1926), co-founder of the modern theory of the triangle and the tetrahedron.[7]

References

1. Nathan Altshiller-Court, *Modern pure solid geometry*, pp. 230. ff. Macmillan, 1935. This book will be referred to as MPSG.

2. MPSG, p. 245, art. 753.
3. „ p. 203.
4. *National Mathematics Magazine*, vol. 15 (1941), p. 274.
5. MPSG, p. 72, art. 238.
6. MPSG, p. 239, art. 735.
7. *Mathesis*, 1922, p. 406.

GLEANINGS

The fundamental distinction between inspired music, chess and mathematics on the one hand and professions, industry and politics on the other is that while the former manifests in the subject without the subject's volition the latter is the direct product of the will and desire of the person concerned. Ramanujam did not desire to become a mathematics prodigy; nor did, Reshevsky or to come near home, our Balachandran exert himself to become a chess genius. But Henry Ford and F. D. Roosevelt are the products of a desire and will to forge ahead in a practical world with all that the process implies. From this it is but one step to infer that while inspired music, mathematics and chess descend upon man—the incarnated soul—from extra-mundane orders of Reality, professions, politics and industry are gathered from the world by the man successful in these respectively. Viewed in this light it is a left-handed compliment to the native chess genius of the Cuban master to say that had he but diverted his attention to 'practical' spheres of life, he would have shone as well as he did in chess.

T. A. KRISHNAMACHARIAR in *The Hindu*.

An Indian Philosopher, being asked what were, according to his opinion, the two most beautiful things in the Universe answered: The starry heavens above our heads, and the feeling of duty in our hearts.

BOSSUET.

ON SOME PROBLEMS OF TRANSPORT ECONOMICS

BY

H. E. PERIES, *Secretariat, Colombo (Ceylon).*

Here we deal with passenger transport between two consecutive termini, the fares being "fixed". In the first section, we ignore the time factor and suppose the fare p is constant. In the next section we deal with the case in which the fare is a step-function of the time $p(t)$.[1]

We suppose that to each fare or fare-function, there is correlated a passenger offering function representing the number of passengers wishing to travel at this fare. In the first section, we treat this as a number $f(p)$ while in the second section we consider it as a distribution in time. The permissible fares are the integral multiples of the least unit of currency and the passenger offering function takes integral or, if we consider averages, rational values.[2] The problem, so limited, is of little interest mathematically. We therefore assume that p can vary continuously. We suppose, also that the passenger offering function has a suitable degree of "smoothness" and derivatives up to the second order, and further that the transport conditions reproduce themselves exactly after a certain period T and we use the parameter t to express the position of an instant in this period.[3] We also assume the existence of a cost function which is also a step-function. The operator is supposed to adjust the variables at his command so as to maximize his profit or the difference between

[1] Vide R. G. O. Allen. *Mathematical analysis for Economists* page 130 Ex. 10 and 11. This paper may be said to deal with the so called Dynamical Problem (*c. f.* for instance J. R. Hicks "Wages and interest, The Dynamical Problem." *Economic Journal* Vol. XLV Sept. 1935.) inasmuch as our problem involves time. It is however characteristic of transport problems that the cost function depends on the distribution of traffic in time. We cannot, also, consider "staggered" fares without considering the time factor. In this sense a static discussion cannot be realistic.

[2] *c. f.* Allen *loc. cit.* pg. 110-112. The range of values is further extended if one supposes that the real value of the currency unit varies with time.

[3] In a static investigation of a long term process, we really assume that an instantaneous section in time is completely representative and that we need consider no factors outside this section our assumption of perfect periodicity means than any section in time of length T is exactly similar to any other section. Our problem is, in this sense, intermediate between the static and dynamic problems, in which no such assumption is made.

the product of the fare and the passenger offering (or in the second section the corresponding integral) and the cost function.

1. In this section we neglect the time factor and suppose that the cost function R depends only on the number of passengers x transported but that $R(x)$ may be discontinuous in X at an enumerable set. We deal with the case of a single operator or monopolist. The profit $P = px - R(x) = p \cdot f(p) - R\{f(p)\}$.

The only difficulty in maximizing this function is that R may have discontinuities and the normal method is therefore not applicable. With a known $f(p) + R$ all we have to do is to compare the maxima, found in the usual way[1] in the enumerable set of continuous stretches, with the values at the ennumerable set of discontinuities. If this operator runs in competition with some other form of transport with fares fixed well in advance, a say, the same reasoning applies with the parametric function $f_a(p)$ replacing $f(p)$.

2. We again consider a monopolist but we no longer suppose p a constant but a step-function of the time t. We metricize the space of the step-functions p by assigning the distance-function

$$\rho\{p_1(t), p_2(t)\} = \int_0^T |p_1(t) - p_2(t)| \cdot dt.$$

It is clear that this function satisfies the three requirements of a distance function.[2] We also consider two capacity functions $N(t)$ and $N'(t)$ for the two termini. These functions specify the time of departure of the buses from the termini (we are mainly concerned with the times of departure as we shall see) and the seating capacity of the departing vehicles. In the interval between consecutive departures, we suppose $N(t)$ and $N'(t)$ to be equal to their values at the next departure, thus making the two functions step-functions. We metricize these two spaces in the same way as for $p(t)$.[3] For the composite space $\{p(t), N(t), N'(t)\}$ we take the square root of the sums of the squares of the distances between the co-ordinates.

[1] *c. f.* Allen *loc. cit.* § 8. 7.

[2] *c. f.* C. Kuratowski *Topologie I* pp. 82, 83; or Sierpinski: *General Topology* chap. VI. If we restrict the values of p to rational values, it is clear that the number of such functions is $\leqslant c^r = c$ where c is the number of points in a linear continuum, and r the cardinal number of an enumerable aggregate. On the other hand by considering the step-functions $p(x) = 0$, $x < a$, $p(x) = 1$ $x \geqslant a$, we see that there are certainly c such functions. There are thus c step-function taking rational values.

[3] A similar argument shows that there are c functions $N(t)$, $N'(t)$, $f(t)$ and $f'(t)$.

We denote by $f(t)$ and $f'(t)$ the two functionals depending on $p(t)$, $N(t)$ and $N'(t)$ which represent the passenger offering in the two directions.[1]

The cost function $R(t)$ is assumed to be a functional of the $N(t)$ and $N'(t)$ only.

The gross takings are

$$\sum_{1}^{n} p(t_r)\cdot\left[\min\left\{\int_{t_{r-1}}^{t_r} f(t)\,dt\right\},\ N(t_r)\right] + \sum_{1}^{m} p(t'_r)\left[\min\left\{\int_{t'_{r-1}}^{t'_r} f'(t)\,dt\right\},\ N'(t'_r)\right]$$

where $t_1, t_2 \cdots t_n$ and $t'_1, t'_2 \cdots t'_m$ are the times of departure at the two termini.[2]

The profit is the difference between the sum and $R(N, N')$. To find the maximum of this we have to use variational methods,[3] the $p(t)$ and the two capacity functions being the adjustable variables.

The method can be extended to case where there are several different fare functions for different categories of passengers and more than one stage. Our new variable step-functions will be the set

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1 We work with a passenger offering function. This causes difficulties in "smoothing" and also in allowing for excess passengers left behind due to overcrowding of a departing bus. These difficulties are avoided if we consider functions $F(t)$, $F'(t)$ representing the number of persons awaiting conveyance in the two directions at the time t we need no longer suppose these continuous.

We could also replace our $N(t)$, $N'(t)$ by total capacity functions $M(t)$, $M'(t)$ step-functions representing the total seating capacity of buses leaving in each direction up to but excluding t. The expression for the gross takings can then be written

$$\int_0^T p(t)\,[F(t)]_{M(t+0)-M(t-0)}\,dM + \int_0^T p(t)\,[F'(t)]_{M'(t+0)-M'(t-0)}\,dM$$

where the square bracket indicates that the function is limited to $M(t+0)-M(t-0)=N(t)$ (*c. f.* de la Vallée Poussin *Integrales de Lebesgue, Fonctions d'ensemble, classes de Baire* 1916 ed. p. 45).

This integral need not exist in the Riemann sense as $p(t)$, $F(t)$ and $M(t)$ can have common discontinuities but it must exist in the Moore-Pollard sense (not generalized) with suitable conventions regarding the sense of the discontinuities of $p(t)$, $F(t)$ and $M(t)$. *c. f.* L. C. Young. An inequality of the Holder type, connected with Stieltjes Integration *Acta Mathematica* Vol. 67 pg. 263 and Love and Young Fonctionelles Lineairies *Fundementae Mathematica.* Tome XXVIII.

2 We could easily adjust our method to the case where different fares are charged in the two directions.

3 *c. f.* Allen *loc. cit.* chap. 20. De la Vallée Poussin *Cours d'Analyse* chap. X See also *Methodes Topologiques dans les Problemes variationnels* by L. Tusternik and L. Schnirelmann.

We use our metric merely to define a neighbourhood of our maximizing step-function. Any metric will give the same result as our problem requires an absolute and not merely relative maximum.

of the corresponding $\{p(t)\}$ for the categories and stages and the set of capacity functions for the various possible combinations of stages.

The adjustment of the method to the case of competition by the Railway is obvious.

We now consider the case[1] of n operators plying between the same two consecutive termini. For simplicity, we limit ourselves to the case analogous to section, in which the time factor can be left out of consideration. We suppose the existing fares have been distributed and we wish to determine the new fares after the period of readjustment. We denote by $x_1, x_2 \cdots x_n$ the number of passengers transported by each and suppose that

$$x_r = f_r(p_1, p_2 \cdots p_r \cdots p_n) \qquad (r = 1 \cdots\cdots n.)$$

Let $R_1(x_1) \cdots\cdots R_n(x_n)$ be the corresponding cost of operation. Each transporter seeks to select that p_r which will maximize his profits but his decision will depend on the fares which he supposes the others will charge. Let us denote by $p_j{}^r$ the fare which the rth operator thinks the jth operator will charge. Assigning these values to the $p_1, p_2 \cdots p_{r-1}, p_{r+1} \cdots p_n$ in f_r, we could determine the value of p_r which would maximize the rth operator's profit by the methods of §1. Let us suppose this method gives us the n equations

$$\phi_r(p_1{}^r, p_2{}^r \cdots p_r \cdots p^r{}_n) = 0 \qquad (r = 1, 2 \cdots n) \qquad \ldots \text{(A)}$$

For equilibrium a necessary condition is that the final fares $p_1, p_2 \cdots p_n$ satisfy the set of equations (A) for otherwise one or more operators would increase his or their profit by changing his fare. This gives us n equations to determine these n quantities.[2]

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[1] *c. f.* Allen, *loc. cit.* § § 8·8 and 13·9 for the duopoly problem treated on the basis of "conjectural variations." It does not, however, deal with the further reactions of the duopolist when he finds his conjecture to be incorrect.

[2] If we suppose the "ϕ"s refer to the relations obtained in section 2, the same relations remain formally correct when the "p"s are considered as step-functions.

A DIALOGUE

BY

PROF. F. W. LEVI, *Calcutta University.*

Student. The problem enunciated at the beginning of this chapter seems to be a very easy one, but I have seen such words as "vectorspace," "rank," "matrix" later in the book; I also noticed determinants and formulas with upper and lower indices. I cannot understand why the author is trying to make a very simple thing so complicated. The problem can be solved with the help of methods which I learned, when I read for the matriculation examination.

Tutor. Of course, it is my duty to help you to understand this theory clearly, but Mathematics is not a matter of seniority. History shows several examples where mathematicians were superior to their masters at a very early age of life. I should not miss the opportunity to learn something from you; please, explain your solution of the problem.

St. It is simply the method of substitution; From the last equation it follows that $x=(k-k_1 x_1-\cdots-k_{n-1} x_{n-1}) \div k_n$. Putting this value into the remaining linear equations, I get linear equations with $n-1$ unknown only. After having solved this system, we calculate the value of x_n by putting the values of $x_1, \cdots, x_{n-1}$ in that equation. Is it so?

T. Yes — provided $k_n \neq 0$.

St. In the case $k_n=0$, x_n is infinite.

T. I do not think so;—*e g.* Consider two equations and $n=2$, $k_2=0$, say $x_1+2x_2=5$, $x_1+0x_2=1$.

The only solution of this system is obviously $x_1=1$, $x_2=2$.

St. Yes, that is true.—If $k_n=0$, then I take another equation of the system in place of the last one. Thus without loss of generality, I suppose $k_n \neq 0$. I think you will be satisfied.

T. Unless the coefficient of x_n is zero in each equation of the system.

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* [The Socratic method of eliciting hidden difficulties by means of a dialogue is pleasant and profitable, and of considerable pedagogic interest. We have great pleasure in reproducing the following dialogue from Prof. Levi's recent publication "Algebra, vol. I" with the kind permission of the Calcutta University Authorities and of the Author—*Ed.*]

St. In this case the system has to be considered as a system with $(n-1)$ variables only; it would be absurd to consider it as a system of n variables as the equations are actually independent of x_n.

T. Perhaps less absurd than you may believe, but I accept your definition that a system of linear equations should be considered to depend on such variables only, as have at least one coefficient different from zero.

St. Certainly.

T. After x_n has been eliminated, how do you continue?

St. I shall repeat the process again and again until I get one equation with one variable x_1, and then there is no problem left.

T. You suppose that the number of equations is equal to the number of variables, and you believe that at every step of your procedure, both the numbers decrease by exactly one?

St. Certainly, but the number of the equations may also be less than the number of the variables, let us say $m < n$ equations in n variables. In this case one puts the term with $x_{m+1}, \cdots, x_n$ to the right hand side. These variables may take arbitrary values. For every set of values $x_{m+1} \cdots, x_n$, there exists one solution $x_1, \cdots, x_r$ as there are as many of these variables, as there are equations. The number $n-m$ is the *degree of freedom* of our system as the values of $n-m$ variables may be chosen arbitrarily.

T. And if there are more equation than variables?

St. Then there cannot exist any solution. It is obvious that n variables cannot satisfy a system of more than n conditions.

T. But it seems to me that the system $x=1$, $2x=2$ has a solution although it is a system of two equations with one variable.

St. But these equations are not different. Equations which differ by a common factor only cannot be considered as different, and it is commonsense to consider only such equations which are different.

T. Thus, if two equal equations are given, one of them should be dropped.

St. Yes.

T. But this must be done also at the later stages of the procedure,

St. I cannot follow you exactly.

T. Consider: $7x_1 - 38x_2 + 3x_3 = 13$

$$3x_1 + 13x_2 + 2x_3 = 17$$

$$2x_1 - 5x_2 + x_3 = 6$$

3 different equations in x_1, x_2, x_3. Since the degree of freedom is zero, do you expect to get exactly one solution?

St. Yes; Put $x_3 = 6 - 2x_1 + 5x_2$ in the first two equations, then

$$x_1 - 23x_2 = -5$$

$$-x_1 + 23x_2 = 5.$$

T. These equations differ by a factor -1 only, hence one of them must be dropped. Thus you may chose x_2 in an arbitrary manner. Put $x_1 = 23x_2 - 5$, $x_3 = -41x_2 + 16$, and this will solve the system of equations for every value of x_2. You have one "degree of freedom," although the number of the equations is equal to the number of the variables.

St. That is true. This example is obviously a wicked exception.

T. You may call it an exception if you like, but there are a plenty of them.

St. I see;—There may be a certain cases, where the degree of freedom is higher than the difference between the number of the variables and the number of the equations, but at any rate n equations with n variables have at least one solution which can be found by the method of substitution.

T. Why?

St. The number of the equations can decrease, as one may get two equations by the procedure of substitution, and then one of them must be dropped, but the number of the variables cannot.

T. Try: $9x_1 - 15x_2 - 3x_3 = 13$

$$3x_1 + 10x_2 + 2x_3 = 1$$

$$2x_1 - 5x_2 - x_3 = 2.$$

St. Substitute $x_3 = 2x_1 - 5x_2 - 2$ in the first and the second equation.

Hence $3x_1 = 7$

$7x_1 = 5.$ This is funny.

T. Indeed, the coefficients of x_2 in both equations become zero, thus the equations have to be considered as equations of one variable only. Now x_1 should be equal to $7:3$ and to $5:7$, that is impossible.

St. Perhaps I was somewhat rash in conceding that a system of equations in n variables should be considered as system in $(n-1)$ variables if the coefficient of one of the variables are all equal to zero. Let us retain x_2 and put $3x_1 - 0x_2 = 7$
$7x_1 + 0x_2 = 5$. Hence $x_1 = 5/7 - 0x_2$, and therefore $0x_2 = 34:7$, $x = \infty$.

T. What do you mean by ∞?

St. Infinity; that is a number which is greater than every other number and equal to $1:0$.

T. Can you calculate with this ∞ as with an ordinary number?

St. Certainly.

T. Then $-\infty = -1:0 = 1:(-0)$ and since $-0=0$, $-\infty = \infty$ holds. Hence $0 = 2\infty$, and therefore $0 = \infty$.

St. No, that is not so. One cannot calculate with this symbol as with an ordinary number. But, as a matter of fact, this ∞ occurs in mathematics. It is somewhat complicated matter, one needs differential calcus to handle it, and I was hoping that you may explain it to me clearly one day.

T. On another occasion. The symbol ∞ does occur; sometimes it is used rightly, sometimes wrongly. Use and misuse, both are found in text books. Considering the systems of linear equations, we enquire about those numbers which taken for $x_1, \cdots, x_n$ respectively, satisfy those equations. Numbers can be added, subtracted and multiplied; one can also divide a number by a number, unless the divisor is zero. The division by zero is meaningless as far as numbers are concerned.

St. And the example which I attempted just before?

T. It has no solution, since x_1 cannot simultaneously be equal to $7:3$ and equal to $5:7$.

St. So there exist systems of 3 equations in 3 variables which have no solutions, systems which have an infinity of solutions, and systems which have exactly one solution. How did you construct those examples by which you cornered me.

T. It is not difficult if one knows a little bit of theory.

St. Those vectorspaces, matrices, rank etc.?—Sir, I should be thankful if you could explain to me some portion without using those notions. I do not like those innovations.

T. Then try the simplest case; $ax=b$.

St. Then $x=b:a$.

T. Provided $a \neq 0$.

St. If $a=0$, $b \neq 0$, the equation has no solution as there exists no number x, for which $0x=b \neq 0$. If however $a=0$, $b=0$, then every value x is a solution.

T. Indeed; This simple case is the seed of the whole theory.

Now try.
$$a_1x+a_2y=a$$
$$b_1x+b_2y=b.$$

St. $\Delta y=\Delta_1$, $\Delta x=\Delta_2$, where $\Delta=a_1b_2-b_1a_2$, $\Delta=a_1b-b_1a$, $\Delta_2=ab_2-ba_2$. If $\Delta \neq 0$, then $x=\Delta_2:\Delta$, $y=\Delta_1:\Delta$.

T. In this case there exists no other solutions, and these values satisfy the given equations, as you may verify easily.

St. If $\Delta=0$, but Δ_1 or Δ_2 is different from zero, there is no solution. If $\Delta=\Delta_1=\Delta_2=0$, then every couple of values (x, y) satisfies the equations.

T. Consider $x+2y=5$, $3x+6y=15$. Here $\Delta=\Delta_1=\Delta_2=0$, but *e.g.* $(x, y)=(0, 0)$ is not a solution, as $x=5-2y$.

St. This is true, but I cannot understand it. The equation $\Delta x=\Delta_2$, $\Delta y=\Delta_1$ are satisfied by every pair of values x, y if $\Delta=\Delta_1=\Delta_2=0$'

T. These equations are consequences of the given equations, they are *necessary* conditions for solutions x, y of the given systems but they may not be *sufficient* ones. The case $\Delta=\Delta_1=\Delta_2=0$ contains different cases.

(1) If all the coefficients are equal to zero, then every pair (x, y) is a solution.

(2) Let $a_1=a_2=b_1=b_2=0$, but $(a, b) \neq (0, 0)$ then there is no solution.

(3) Let at least one of the coefficients on the left side be$\neq 0$. Without loss of generality, $a_1 \neq 0$, Put $b_1 : a_1 = \lambda$.

Hence, $$0 = \Delta = a_1(b_2 - \lambda a_2),\ b_2 = \lambda a_2$$
$$0 = \Delta_1 = a_1(b - \lambda a),\ b = \lambda a.$$

$(b_1x + b_2y - b) = \lambda(a_1x + a_2y - a)$. Hence $x = (a - a_2y) : a_1$ for arbitrary y furnishes all the solutions. There are therefore five different cases.

St. For a higher number of variables and of equations a full analysis may become very complicated. How to tackle the problem for an arbitrary n?

T. For this, I propose to you to study the notions of vector-space, rank, matrix, determinant etc., as explained in the following articles.

GLEANINGS

Marcellus, though at a loss what to do, and not knowing how to oppose the machines of Archimedes, could not, however, forbear jesting upon them. "Shall we persist," said he to his workmen and engineers, "in making war against this Briareus of a geometrician, who treats my galleys and sambucas so rudely? He infinitely exceeds the fabled giants, with their hundred hands, in his perpetual and surprising discharges upon us." Marcellus had reason for complaining of Archimedes alone. For the Syracusans were really no more than members of the engines and machines of that great geometrician, who was himself the soul of all their powers and operations. All other arms were unemployed; for the city, at that time, made use of none, either defensive or offensive, but those of Archimedes. Marcellus at length, perceiving the Romans so much intimidated, that if they saw upon the walls only a small cord, or the least piece of wood, they would immediately fly, crying out that Archimedes was going to discharge some dreadful machine upon them, renounced his hopes of being able to make a breach in the place, gave over his attacks, and turned the siege into a blockade. * * * Deprive Syracuse of only one old man, the great strength of the Roman arms must take the city.

From ROLLIN: *History of the Ancients.*

ANOTHER DIALOGUE †

By

Prof. F. W. Levi, *Calcutta University.*

Student. When I started reading Algebra, you advised me to study carefully the systems of linear equations. So I did. I further read general Algebra and continued fractions. At first it was hard work, but later on, I was quite successful.

Tutor. Alright, but I do not think you have met me for this. You are looking rather despondent.

Stu. Indeed, I am again in the wilderness.

Tu. Why, is there some difficulty in the book which you want me to explain?

Stu. It is not for that, but the whole subject became problematical to me again.

Tu. I wonder how.

Stu. Yesterday an engineer asked me to solve a certain algebraic equation. I replied that, in consequence of the fundamental theorem of General Algebra, there exist roots in a suitable extension of the field of the coefficients, and that from the fundamental theorem of classical Algebra, this extension can be chosen as the field of the complex numbers. Thus there exist complex roots of the equation and some of them might be real.

Tu. Was the engineer satisfied by your reply?

Stu. Not at all; he said that I seemed to be a great philosopher, and that I had missed the point completely. He was interested in real roots only. And he had no doubt about their existence. He has found out that the force (expressed in kilogrammes) acting on a certain part of an engine was bound to satisfy that equation. He was asking me to compute that force and nothing else.

Tu. And you could not. The polynomial was too complicated.

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† [This is also from Prof. Levi's *Algebra* and is published with the permission of the author and the Calcutta University Authorities.—Ed.]

Stu. It looked very simple. It was something like this x^5+4x^3+2x+6. From Eisenstein's theorem it is irreducible and therefore its real roots must be irrational. This I told the engineer.

Tu. Perhaps the good man did not know anything about irrationality.

Stu. He did! But he was not at all interested in my statement. He said, "I do not want to have an infinity of decimals, even if you can provide me with them. Compute the kilogrammes; I leave the grams etc. to you." Now, for any positive x the polynomial takes positive values only. So, I told him that the real roots of the polynomial must be negative.

Tu. And, was this statement of any use to the engineer?

Stu. No, he knew already that the force was directed to the negative side; and then he said. 'The direction of the force is not very interesting to me, as there is little difference whether the material is exposed to stress or to pressure. If you give me a solution with 30% of error and a wrong sign, I could make some use of it. But your philosophical talk is worth nothing." He was quite rude eventually.

Tu. And you?

Stu. I am bewildered. After having read about 200 pages of the book, I am still unable to solve a very simple algebraic problem, not even if 30% of error and a wrong sign are admitted. Though I got very interested in Algebra, the Engineer's argument has impressed me. I am afraid that all my hard work has seen spent uselessly.

Tu. I rather think, you have stopped reading at the wrong place. If you continue, you will be able to provide your friend with a solution, with considerably less than 30% of error.

Stu. I had already a glance on the next chapter but I do not see any connection between its content and the preceeding parts of the book (eg.) general Algebra; and then, there is another thing that strikes me, every solution is given only approximately. I should like to know the solutions correctly. If for a particular application, a few decimal places only are requested, then I may neglect the higher terms of the correct result. But as a student of Pure Mathematics, I must know at first the proper solution, before admitting some error for the sake of abbreviation.

Tu. How do you want to represent the solution if it happens to be an irrational number?

Stu. There are many ways of expressing irrational numbers. For instance $\sqrt{2}$ is irrational. It cannot be expressed as a ratio of 2 integers, but nevertheless, $\sqrt{2}$ is a number. Everybody knows what is $\sqrt{2}$.

Tu. Suppose that I do not know and you try to explain.

Stu. $\sqrt{2}$ is the positive number whose square is equal to 2.

Tu. Well, I take it for granted that one and only one such positive number exists. Let x be positive and $x^2=2$. Then $x=\sqrt{2}$, or $\sqrt{2}$ is the positive root of $x^2-2=0$. I think this statement is completely equivalent to yours.

Stu. It is.

Tu. You seem to be satisfied with this manner of expressing irrational numbers.

Stu. Of course, I am. If I could express the roots of every algebraic equation in this way, there would be nothing to complain of.

Tu. My point is that in this case, the roots are expressed by a tautology.

Stu. I cannot follow you.

Tu. Listen, which are the roots of the polynomial x^2-2?

Stu. $\sqrt{2}$ and $-\sqrt{2}$.

Tu. Whereby $\sqrt{2}$ is nothing else than a symbol for the positive root of x^2-2. Besides the statement, that x^2-2 has two real roots and that their sum is equal to zero, your solution of the problem to find the roots of x^2-2 is a mere tautology. Your conclusion goes like this: "Who is Amal?" "The Brother of Bimal"— "And who is Bimal?" "Amals' brother." That means only that there exist two brothers Amal and Bimal, but it does not explain who is Amal.

Stu. But $\sqrt{2}$ is a well-known number; Mathematicians have got used to it. and they calculate with $\sqrt{2}$ as they do with 23 or 1 : 7. For me, there is no problem about $\sqrt{2}$.

Tu. Is it for the symbol $\surd$, that you hold this opinion?

Stu. $\surd$ as a symbol is a mere convention, the mathematicians could use any other notation instead of it, but I do not see any reason why symbols familiar to everybody should be replaced by new ones.

Tu. I fully agree with you, but for the sake of our conversation let us denote the real roots of a polynomial $f(x)$, as far as they exist, by $[f(x)]_1$, $[f(x)]_3$, $\cdot \cdot \cdot \cdot$ $[f(x)]_n$ in their order of magnitude starting with the greatest root.

Then your explanation of $\surd 2$ simply means that $\surd 2 = [x^2 - 2]_1$.

Stu. Now you want me to admit that for every $f(x)$ which has a real root, the symbol $[f(x)]_1$ must be considered as a solution of the equation $f(x) = 0$. You propose that there is no higher justification in considerating $\surd 2$ as the given number than e. g. $[x^6 + 4x^5 + 2x + 6]_1$. But there is a huge difference between these two cases.

Tu. How is that?

Stu. We know more about $\surd 2$ than that it is positive and that its square is equal to two.

Tu. What do you know about $\surd 2$?

Stu. $\surd 2 = 1{\cdot}4142 \cdot \cdot \cdot \cdot$

Tu. $1{\cdot}4142$ is a rational number, whereas $\surd 2$ is irrational.

Stu. Certainly, it is an infinite decimal fraction, but they have computed 200 decimals or even more of them. You cannot deny that $\surd 2$ is very well known.

Tu. There still remains a certain error.

Stu. But a negligible one.

Tu. That depends on the purpose of the calulation. I was told that certain students of Pure Mathematics must know the proper solution before admitting some error for the sake of abbreviation, was it not so?

Stu. But $\surd 2$ is uniquely determined as the only positive root of $x^2 - 2$.

Tu. Yes there exists one and only one such root. This is a statement on existence and on uniqueness but nothing more than that.

I think we have agreed already about this item. On the other hand, I admit that we know more than that about $\sqrt{2}$. For instance approximately equal to $1{\cdot}4142\cdots$. Or to put it more clearly, $\sqrt{2}$ lies between $1{\cdot}4142$ and $1{\cdot}4143$. One can find out easily small intervals where $\sqrt{2}$ is situated. There is no limit to the improvement of the approximation, and the diminution of the error. This error is not a kind of mistake which is the result of a negligent treatment. On the contrary it is an essential part of the solution of the problem. One cannot *determine irrational numbers otherwise than approximately*. This fact is concealed by symbols we are using. Numbers represented by them are uniquely determined in the sense that there exists one and only such number but one cannot determine the place of an irrational number on the real axis otherwise than approximately.

Stu. Thus a formula like $\sqrt[3]{14+\sqrt{170}}$ is only a recipe how to determine an irrational number approximately.

Tu. The formula denotes the greatest root of $x^6-28\,x^3+26$ and it shows of course a recipe how to compute that number approximately. A mental calculation furnishes 3 as a first approximation which would satisfy your friend completely.

Stu. Suppose one could represent every root of a polynomial by the help of similar symbols. This would furnish a recipe to determine every root approximately.

Tu. As a matter of fact, not every root is representable in that manner, and even if it is, one prefers a different method sometimes.

Stu. My impression is that those methods have no connection with General Algebra. Theory of Approximation and General Algebra *apparently belong to different* branches of Mathematics if not to two different sciences.

Tu. They are complementary to each other. You already mentioned the two fundamental theorems which state the existence of roots in certain fields. They must be supplemented by investigations about where the roots are situated in the field. The methods of investigation must tally with the structure of the particular field under consideration; they cannot be of a general nature. The real numbers are ordered linearly, whereas the complex numbers correspond to the points of a plane. Hence one subdivides the real axis into intervals to determine the real numbers and

similarly the plane is subdivided into certain domains [eg. rectangular or circular ones] to locate complex numbers. Both ways lead to an approximate determination of number (eg.) roots of a polynomial.

Stu. As the n roots of the polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ are uniquely determined by the numbers $a_0 \cdots a_{n-1}$, there must exist functions $f(a_0 \cdots a_{n-1})$ which show the distribution of roots in the complex plane. One should investigate these functions; this would be a worthy continuation of General Algebra.

Tu. If you take the word function in the most general sense, there exist indeed such functions $f(a_0 \cdots a_{n-1})$ *e. g* the set of the roots itself is one. The problem is how to represent these functions; you should not expect that all of them are polynomials in $a_0 \cdots a_{n-1}$. Every polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ can be represented by a point $P = (a_0 \cdots a_{n-1})$ of an n-dimensional space. There are theorems stating that if certain inequalities in $(a_0 \cdots a_{n-1})$ hold *i. e.*, if P is situated in a particular domain of the n-dimensional space, the n roots are distributed in the complex plane in a particular manner. These investigations are very interesting, but at the present time, the approximation of the roots is based more on the methods of calculation than on these theorems. In many cases, the theorems seem to be the result of the practice of calculation. For this reason, the author has started from Horner's scheme which gives the clue to the whole theory. I advise you to work out many numerical examples, it will help you to understand the theoretical portion!

DEATH OF ARCHIMEDES

Archimedes, at a time when all things were in this confusion at Syracuse, shut up in his closet like a man of another world, who has no regard for what is passing in this, was intent upon the study of some geometrical figure, and not only his eyes but the whole faculties of his soul were so engaged in this contemplation that he had neither heard the tumult of the Romans universally busy in plundering, nor the report of the city's being taken. A soldier, on a sudden, comes in upon him and bids him follow him to Marcellus. Archimedes desired him to stay a moment, till he had solved his problem and finished the demonstration of it. The soldier who neither cared for his problem or demonstration, enraged at this delay, drew his sword and killed him. Marcellus was exceedingly afflicted when he heard the news of his death. Not being able to restore him to life, of which he would have been very glad, he applied himself to honour his memory to the utmost of his power. He made a diligent search after all his relations, treated them with great distinction and granted them peculiar privileges. As for Archimedes, he caused his funeral to be celebrated in the most solemn manner and erected to him a monument among the great persons who had distinguished themselves most at syracuse.

From ROLLIN: *History of the Ancients*.

NOTES AND DISCUSSIONS

Metrization by means of a function satisfying a single axiom.

1. The object of this note is to show that if, for every pair of elements x, y of a set, a real-valued function $\rho(x, y)$ is defined satisfying the axiom

$$\text{(A)} \qquad \rho(x, z) + \rho(y, z) \geqslant \rho(x, y) + \rho(z, z)$$

then the set can be metrized. This is done by constructing a function $d(x,y)$ in terms of $\rho(x, y)$ which shall satisfy the metric axioms

$$\text{(B)} \quad B_1: \quad d(x, x) = 0$$

$$B_2: \quad d(x, y) = d(y, x)$$

$$B_3: \quad d(x, y) + d(y, z) \geqslant d(x, z).$$

Lemma 1: $\rho(x, y) = \rho(y, x)$.

Proof: Putting $z = x$ in (A) we get $\rho(y, x) \geqslant \rho(x, y)$, and from symmetry $\rho(x, y) \geqslant \rho(y, x)$, whence the results follows.

Lemma 2: $\rho(y, z) \geqslant \frac{1}{2}[\rho(y, y) + \rho(z, z)]$.

This is easily proved by putting $x = y$ in (A).

2. Now set $d(x, y) = \rho(x, y) - \frac{1}{2}[\rho(x, x) + \rho(y, y)]$.

By lemma 1 it is easily seen that $d(x,y)$ satisfies axioms B_1 and B_2. That it satisfies axiom B_3 is proved as follows:

$$d(x, z) + d(z, y) = \rho(x, z) + \rho(z, y) - \rho(z, z) - \tfrac{1}{2}[\rho(x, x) + \rho(y, y)].$$

$$\geqslant \rho(x, y) - \tfrac{1}{2}[\rho(x, x) + \rho(y, y)]. \text{ by lemma 1 and}$$

axiom (A).

$$= d(x, y).$$

For a metric space, in addition to the axioms B we require the axiom

$$B_4: \quad d(x, y) > 0 \text{ if } x \neq y.$$

This is achieved by strengthening lemma 2 by the axiom

$$\text{(A')} \quad \rho(x, y) > \tfrac{1}{2}[\rho(x, x) + \rho(y, y)] \text{ if } x \neq y.$$

3. It is obvious that all distance functions satisfy axiom (A). But there are also functions which are not distance functions and which satisfy axiom (A). One trivial example is the function

$$\rho(x, y) = d(x, y) + a,$$

where d is a distance function and a is some constant. It is not easy to find many concrete illustrations of sets wherein a function ρ may be defined which is neither a distance function nor one of the trivial type given above.

The following examples may be of some interest.

Example 1. Consider the set of all real numbers and let $\rho(x,y)$ be defined to be the greater of x and y. ρ satisfies axiom (A) since

$$\max(x, z) + \max(y, z) \geqslant \max(x, y) + z,$$

The distance function defined in terms of ρ is

$$d(x, y) = \max(x, y) - \tfrac{1}{2}(x + y)$$
$$= \tfrac{1}{2}|x - y|.$$

Example 2. Consider the set of all positive integers and let the l.c.m. and g.c.d. of two numbers a, b be denoted by $[a, b]$, and (a, b) respectively. Since the equivalent relations

$$[a, c]\,[b, c] \geqslant c\,[a, b]\ ; \quad \frac{1}{(a, c)(b, c)} \geqslant \frac{1}{c(a, b)}$$

always hold we may take $\rho(a,b) = \log\,[a,b]$ or $\rho(a,b) = \log\dfrac{1}{(a,b)}$. The ρ's so defined will both satisfy axiom (A) and they give rise to the distance function

$$d(a,b) = \log\frac{[a, b]}{\sqrt{a\,b}} = \log\frac{\sqrt{a\,b}}{(a, b)} = \tfrac{1}{2}\log\frac{[a, b]}{(a, b)}.$$

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On Conformal Transformations

Many complete criteria have been given for the conformality of a plane transformation, all amounting, of course, to the Cauchy-Riemannian conditions directly or indirectly. I add one more below, assuming that the transformation is reversible. Conformality is here viewed in terms of the parametric curves in the original and the image planes and my object is to give a property of these curves which corresponds fully to the conformal property. The assumption that the inverse transformation exists is made since with it we obtain a very simple property of the parametric curves, *viz* orthogonal intersection, as a complete criterion. As is well known, orthogonal intersection in one of the planes only is insufficient.

THEOREM: "*A necessary and sufficient condition that a reversible plane transformation* $\xi = \phi(x, y)$, $\eta = \Psi(x, y)$ *be conformal is that the transformation and its inverse be both orthogonal. i.e. that the parametric curves in the* (ξ, η) *as well as the* (x, y) *planes do intersect orthogonally.*"

The necessity of the condition follows at once from the fact that if a transformation is conformal, then the reverse transformation also is conformal and every plane conformal transformation is orthogonal.

To prove sufficiency, let the inverse transformation be

$$x = g(\xi, \eta), \quad y = h(\xi, \eta)$$

so that

$$\xi = \phi\{ g(\xi, \eta), h(\xi, \eta) \}$$
$$\eta = \Psi\{ g(\xi, \eta), h(\xi, \eta) \}$$

Differentiating we obtain

$$\left.\begin{aligned} 1 &= \phi_x g_\xi + \phi_y h_\xi \\ 0 &= \Psi_x g_\xi + \Psi_y h_\xi \end{aligned}\right\} \qquad \begin{aligned} 0 &= \phi_x g_\eta + \phi_y h_\eta \\ 1 &= \Psi_x g_\eta + \Psi_y h_\eta \end{aligned}$$

which give the following relations between the partial derivatives:

(i) $g_\xi = \Psi_y/J, \quad g_\eta = -\phi_y/J, \quad h_\xi = -\Psi_x/J, \quad h_\eta = \phi_x/J$

where

$$J = \partial(\xi, \eta)/\partial(x, y) \neq 0$$

Now the parametric curves $g(\xi, \eta) = \text{const.}$, $h(\xi, \eta) = \text{const.}$ intersect at an angle

$$\cos^{-1} (h_\xi g_\xi + h_\eta g_\eta)/\sqrt{(h_\xi^2 + h_\eta^2)}\sqrt{(g_\xi^2 + g_\eta^2)}$$

and, since this is a right angle, we have

(ii) $$h_\xi g_\xi + h_\eta g_\eta = 0$$

Likewise since the curves $\phi(x, y) = \text{const.}$ $\Psi(x, y) = \text{const.}$ intersect at right angles, therefore

(iii) $$\phi_x \Psi_x + \phi_y \Psi_y = 0$$

In virtue of the relations (i), the relation (ii) becomes

(iv) $$\phi_x \phi_y + \Psi_x \Psi_y = 0$$

Solving (iii) and (iv) we obtain

$$\phi_x = \Psi_y \text{ and } \phi_y = -\Psi_x$$

or $$\phi_x = -\Psi_y \text{ and } \phi_y = \Psi_x$$

which are the Cauchy-Riemannian equations. Hence the transformation is conformal.

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A Contact Transformation

In this paper I show that the transformation defined by the equations

(A) $$\begin{cases} X = x - \dfrac{3y''\,y'''}{3y''\,y'''' - 5y'''^2} \\ Y = y + \dfrac{3y''\,(3y''^2 - y'\,y''')}{3y''\,y'''' - 5y'''^2} \end{cases}$$

where the accents denote differentiations with respect to x, is a contact transformation of order four and obtain the corresponding geometrical theorem.

Let the right members of (A) be called f and ϕ respectively. Then we have

$$\frac{dY}{dX} = \frac{\dfrac{\partial\phi}{\partial x} + \dfrac{\partial\phi}{\partial y}\,y' + \dfrac{\partial\phi}{\partial y'}\,y'' + \dfrac{\partial\phi}{\partial y''}\,y''' + \dfrac{\phi\partial}{\partial y'''}\,y'''' + \dfrac{\partial\phi}{\partial y''''}\,y'''''}{\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\,y' + \dfrac{\partial f}{\partial y'}\,y'' + \dfrac{\partial f}{\partial y''}\,y''' + \dfrac{\partial f}{\partial y'''}\,y'''' + \dfrac{\partial f}{\partial y''''}\,y'''''},$$

which, on substitution for the partial derivatives and simplification,* becomes

$$\frac{dY}{dX} = \left(y' - \frac{3y''^2}{y'''}\right)$$

* The factor which is removed is the left member of Monje's differential equation of the conic viz. $(40\,y'''^3 - 45\,y''\,y'''\,y'''' + 9\,y''^2\,y''''')$. This means merely that the curve should be of order > 2.

an expression which does not involve derivatives of order greater than those involved by (A). Again differentiating the last equation with respect to X we get

$$Y'' = \frac{(3y''y'''' - 5y'''^2)}{y'''\,(40y'''^3 - 45y''y'''y'''' + 9y^2\,y''''')}\,\frac{d}{dx}\left(y' - \frac{3y''^2}{y'''}\right)$$

which does not involve derivatives of order greater than or equal to six. It is clear that in general

$$\text{(B)} \qquad Y^{(n)} = \psi\,(x, y, y', y'', \cdots, y^{(n+3)})$$

Thus since $Y^{(n)}$ is independent of $y^{(n+4)}$ and higher derivatives, therefore (A) is a contact transformation of order four.

Now consider the equation

$$\begin{vmatrix} x^2 - x_1^2 & (xy - x_1 y_1) & y^2 - y_1^2 & 2(x - x_1) & 2(y - y_1) \\ x_1 & x_1 y' + y_1 & y_1 y' & 1 & y' \\ 1 & x_1 y'' + 2y' & y_1 y'' + y'^2 & 0 & y'' \\ 0 & x_1 y''' + 2y'' & y_1 y''' + 3y' y'' & 0 & y''' \\ 0 & x_1 y'''' + 4y''' & y_1 y'''' + 4y' y''' + 3y'^2 & 0 & y'''' \end{vmatrix} = 0$$

where y', y'', y''' and y'''' denote derivatives of the curve $y=f(x)$ with respect to x calculated for the point (x, y) on it. This equation, which is obtained by eliminating the quantities a, h, b, g, f from

$$a\,(x^2 - x_1^2) + 2h\,(xy - x_1y_1) + b\,(y^2 - y_1^2) + 2g\,(x - x_1) + 2f\,(y - y_1) = 0$$

and the four equations resulting from this by four successive differentiations with respect to x, gives the osculating conic of the curve $y=f(x)$ at the point (x_1, y_1) on it, the derivatives y', y'', y''' and y'''' at that point being the same for the conic as for the curve. Transferring the origin to the point (x_1, y_1) the coefficients will have the values:

$$a = 3y'^2y''y'''' - 4y'^2y'''^2 - 6y'y''^2y''' + 9y''^4$$

$$h = 4y'y'''^2 + 3y''^2y''' - 3y'y''y''''$$

$$b = 3y''y'''' - 4y'''^2$$

$$g = 9y'y''^3$$

$$h = -9y''^3,$$

whence the co-ordinates of the centre are obtained as

$$\left(\frac{-3y''y'''}{3y''y'''' - 5y'''^2},\ \frac{3y''(3y''^2 - y'y''')}{3y''y'''' - 5y'''^2}\right)$$

Transferring back to former origin, the co-ordinates (X_1, Y_1) of the centre of aberrancy are given by

$$\text{(C)} \qquad X_1 = x_1 - \frac{3y''y'''}{3y''y'''' - 5y'''^2}\ ; \quad Y_1 = y_1 + \frac{3y''(3y''^2 - y'y''')}{3y''y'''' - 5y'''^2}.$$

Now let c_1, c_2 be two curves having contact of order four at a point (x, y) so that the point and the derivatives y', y'', y''' and y'''' at that point are common to them. The transformation (A) will carry them over into the loci l_1, l_2 of their centres of aberrancy, as shown by (C), and these loci will have the point (X, Y), which corresponds to (x, y), in common between them since (A) involves derivatives up to order four only. Further, l_1 and l_2 should touch each other at (X, Y) since the transformation has contact property. In general (B) leads us to the

THEOREM: "*If two curves c_1, c_2 have contact of order n, $n \geqslant 4$, at a point, then the loci l_1, l_2 of their centres of aberrancy will have contact of order $(n-3)$ with each other at the corresponding point.*"

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GLEANINGS

This [graded catapults to throw stones and darts to various distances] was not the greatest danger. Archimedes had placed lofty and strong machines behind the walls, which suddenly letting fall vast beams, with an immense weight at the end of them, upon the ships sunk them to the bottom. Besides this, he caused an iron grapple to be let out by a chain; and having caught hold of the head of a ship with this hook, by means of a weight let down within the walls, it was lifted up and set upon its stern and held so for sometime; then by letting go the chain, either by a wheel or a pulley, it was let fall again, with its whole weight either on its head or side, and often entirely sunk. At other times the machines dragging the ship towards the shore by cordage and hooks, after having made it whirl about a great while, dashed it to pieces against the points of the rocks, which projected under the walls, and thereby destroyed all within it. Galleys frequently seized and suspended in the air, were whirled about with rapidity, exhibiting a dreadful sight to the spectators, after which they were let fall into the sea, and sunk to the bottom with their crew.

From ROLLIN: *History of the Ancients.*

THALES OF MILETUS

He was the first of the Greeks that studied astronomy. He had exactly foretold the time of the eclipse of the Sun that happened in the reign of Astyages, king of Media.

He was also the first that fixed the term and duration of the solar year among the Grecians. By comparing the bigness of the Sun's body with that of the Moon, he thought he had discovered that the body of the Moon was in solidity but the 720th part of the Sun's body and consequently that the solid body of the Sun was 700 times bigger than the solid body of the Moon. This computation is very far from the truth * * * When Thales travelled in Egypt, he discovered an easy and certain method for taking the exact height of the pyramids, by observing the time when the shadow of our body is equal in length to the height of the body itself * * * As he was one day walking, and very attentively contemplating the stars, he chanced to fall into a ditch. "Ha!" says a good old woman who was by, "How will you perceive what passes in the heavens and what is so infinitely above your head, if you cannot say what is just at your feet, and before your nose?"

From ROLLIN: *History of the Ancients.*

REVIEWS

Finite Geometrical Systems

Six public lectures by Professor F. W. LEVI, Dr. Phil. Nat., Hardinge Professor of Mathematics, Calcutta University; Published by the University, 1942.

We welcome this timely monograph published by a premier Indian University, on a subject in which there is visible progress from day to day. It is a fascinating theme—combinatories—the learned professor has chosen to expound, though not without some misgivings that quick results may not be forthcoming from the mathematical investigations to satisfy the statistician's need of incomplete block designs. Briefly put, an incomplete block design is a system of b blocks containing k varieties each, selected out of v varieties so that each occurs with frequency r and each pair of varieties occurs with frequency λ among the b blocks. The mathematician in his aesthetic urge has developed such designs under various circumstances some of which are put lucidly before the reader in half a dozen talks of equal length.

The reader is first introduced to 'regular graphs' consisting of segments joining points red and blue, denoting varieties and blocks respectively. This colourful representation strongly reminds one of the early Indian efforts, notably Bhaskara's, to give the unknown quantities in algebra colour-names, such as *Kalaka*, *Nilaka*, *Lohitaka*, and *Pitaka*. From a theorem on 'even graphs', necessary and sufficient conditions for the existence of symmetric designs are deduced and illustrated by simple examples.

Again these designs are considered as general finite plane projective geometries, every geometry of this kind with its conics (quadrics), etc., furnishing a block design. Galois fields furnish the clue to the actual construction of such geometrical designs in a certain class of cases.

A step forward is taken to review the possible contributions of combinatorial topology. If one considers the points as varieties and the cells as blocks, every regularly subdivided topological surface of α_0 points, α_1 segments and α_2 cells can be considered as a block system, with $v=\alpha_0$, $b=\alpha_2$, $r=a_0$, each point being incident with a_0 segments and each cell with $a_2=k$ segments subject to the conditions

$$a_0\alpha_0=2\alpha_1=a_2\alpha_2,\ \lambda(v-1)=r(k-1),\ K=2-\alpha_0+\alpha_1-\alpha_2\geqslant 0, \text{ and } \lambda\geqslant 2$$

Among the block designs deduced from such surfaces we are glad to note one which has been missed in the tables of Fisher and Yates, viz.,

127, 347, 567, 432, 614, (135), 236
237, 457, 167, 125, 341, (246), 563

corresponding to $v=7$, $k=3$, $b=14$, $r=6$, $\lambda=2$, of which the blocks in

brackets are missing in the text and supplied by the reviewer. It is interesting to note that this design is made up of two similar designs as arranged above in separate rows, the design represented in the second row being simply a duplicate of that in the first row under a different permutation of the digits. Professor Levi submits that the design ($v = 9$, $b = 24$, $k = 3$) corresponding to $a_0 = 8$ is not known to him. However, it may be pointed out that this unknown case has been dealt with partially by R. C. Bose * who says that it can be obtained by duplicating his design $v=9$, $b=12$, $k=3$. But here a warning may be necessary. Mere duplication is rather crude. It will be better to think of a proper substitution operating on the symbols so as to get a new set of blocks having no block in common with the original set. It is possible that such a substitution does not exist.

Sometimes what is topologically impossible gives a satisfactory block design while the principle of topological duality between points and cells is inadmissible in an unsymmetrical block design, for if $v \neq b$ the interchange of v and b on the one hand and r and k on the other will violate the λ-condition. Prof. Levi is right in conceiving the λ-condition as a real stumbling block to the use of topological armoury by the statistician. Graphs are perhaps more serviceable.

The statistician must be grateful to the warning that too much faith cannot be put in topological proofs which may contain an undetected fallacy. For example, the theorem regarding the possibility of Euler squares of order $4n-2$ (n being a positive integer) has no satisfactory proof in Prof. Levi's opinion. But statisticians like F. Yates† seem to believe in the theorem.

The fourth lecture discusses the existence of geometrical designs which have no analogues in two dimensional projective geometry, such as non-Desarguesian geometrical designs, where the straight lines do not possess the familiar Euclidean straightness but may be bent anyhow on account of the relaxing of some condition like the commutative law of multiplication. It is shown that this commutative law of multiplication is a luxury for Desargues's theorem. For non-Desarguesian geometries one can relax the conditions still further by removing the associative law of multiplication and one of the distributive laws also. In this way by progressive grants of enfranchisement almost new species of geometries are ushered into existence, involving, of course, concomitant complexities in mathematics. Suddenly, however, some unforeseen simplifications arise, such as Wedderburn's theorem that every finite skew field is a field; and Desargues' theorem holds in every geometry of higher dimensions than two. The next two lectures take us still further afield to systems of elements based on Abelian groups, non-isomorphic to Galois fields and doubly transitive groups of degree p^n and order $p^n(p^n-1)$ in which the subgroup not displacing any particular object is non-commutative.

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* R. C. Bose (1939) *Annals of Eugenics.*, 9, pt 4, 373, 377.

† F. Yates., Imperial Bureau of Soil Science, Technical Communication No. 35: *The Design and Analysis of Factorial Experiments*, page, 82.

The sixth and the last lecture introduces the reader to alternating fields and rings in which the commutative and associative laws of multiplication are both relaxed while the two distributive laws are held intact. A mild form of the associative law still prevails as $a(ab)=(aa)b$. Miss Ruth Moufang is ruthless in her scrutiny and investigation of the different aspects of Desargues's theorem. Prof. Levi quotes at some length from her researches and the text here becomes a little obscure to the general reader, who is apt to lose patience but for the comforting prospect that the book closes in a couple of pages more. The last announcement in the text is the theorem that a finite alternating ring containing no divisors of zero is a Galois field. The final moral is that there are four types of geometries:

(i) Geometry over any field in which all the incidence theorems of ordinary geometry hold; (ii) geometry over a skew field where Desargues's theorem holds but not Pappus's theorem; (iii) semi-Desarguesian geometry in which harmonic points exist and the theorem of the complete quadrilateral holds, corresponding to an alternating but not skew field; and (iv) geometry in which harmonic points coincide and Desargues's theorem completely fails.

All this feast of geometries is bound to inspire the statistician with new vigour to carry on his own 'combinatories', partaking of such assistance as he may get from the pure mathematician who is interested more in showing where the quarry lies than in actually digging out particular results.

A few errors of omission in print as, for example, in pages 17 and 48, some alien turns of expression like polyga and conics osculating a straight line, the absence of a select bibliography—these are little defects which can be easily set right in a subsequent edition.

Mysore, A. A. KRISHNASWAMI AYYANGAR.

ARCHIMEDES TOMB

Archimedes, by his will, had desired his relations and friends to put no other epitaph on his tomb, after his death, than a cylinder circumscribed by a sphere; and to set down at the bottom the proportion which those two solids, the containing and the contained, have to each other. He might have filled up the bases of the columns of his tomb with relievos, whereon the whole history of the siege of Syracuse might have been carved and himself appeared like another Jupiter thundering upon the Romans. But he set an infinitely higher value upon a discovery, a geometrical demonstration, than upon all the so-much celebrated machines which he had invented.

Hence he chose rather to do himself honour in the eyes of posterity, by the discovery he had made of the relation of a sphere to a cylinder of the same base and height; which is as two is to three.

From ROLLIN: *History of the Ancients.*

Galois Lectures

BY JESSE DOUGLASS, PHILLIP FRANKLIN, C. J. KEYSER AND LEOPOLD INFELD, Scripta Mathematica Library No. 5, New York 1941, 124 pp ; Price $ 1. 25.

The mathematical public owes to the organisers of the *Scripta Mathematica*, not only an attractive quarterly, but also a series of readable booklets intended to give a birds-eye-view of some of the most interesting regions within and on the border of the subject which no mathematical tourist should miss. The booklet under review, is the fifth of the series and contains 4 lectures delivered at the Galois Institute of Mathematics, Long Island University, Brooklyn, N. Y., whence the title of the book.

In the first address Prof. Jesse Douglas gives a Survey of the Theory of integration considering therein the integrals associated with the names of Riemann, Stieltjes, Lebesgue and Denjoy, the last receiving only a brief mention. Criteria for existence and the properties of the integrals are given with simple examples, and improper and multiple integrals also come in for consideration with a reference to their connection with Fourier Series and Transforms. References are given to standard books so that after the "survey", the reader may dwell "happily ever after" in one of these regions.

The next lecture by Prof. Phillip Franklin deals with the intriguing "Four Colour Problem" enunciated in 1850 by De Morgan e.g. that every geographical map on a plane or a sphere can be coloured in 4 colours, so that no two regions which have a boundary in common are coloured alike. Experiments with diverse maps support this result but there is no mathematical "proof" yet. Heawood proved in 1890 that 5 colours suffice, but the reduction, from 5 to 4 is one of extra-ordinary difficulty and has been nibbled at by a large number of mathematicians, and painfully extended till in 1940 Winn proved that for all maps of 35 or fewer regions 4 colours suffice. In other cases, the possibility of colouring with 4 colours can be proved at present only if the map contains certain special configurations such as a region with 4 sides, rings of 4 or fewer regions, a ring of 5 regions not surrounding a single pentagon etc. For being capable of colouration in 2 colours only, like a chess board a necessary and sufficient condition is that through each vertex an even number of edges pass, while for 3 colours the corresponding condition is that each region has an even number of sides. An example of the latter is a set of hexagons arranged like the cells in a beehive.

When we pass on to multiply connected surfaces like the anchor ring, the minimum number of colours is 7, while for a one sided surface like the projective plane the "chromatic number" is 6. In higher dimensions, the problem becomes trivial since even in 3-space there is no limit to the number of colours required. To see this take 2 layers of long square prisms the top layer at right angles to the bottom layer fastening the nth prism of the top layer with the nth prism of the bottom layer and regarding each such pair as a single block. Each block then touches every other and should be coloured differently from the others.

Problems equivalent to the 4 colour problem are also mentioned. For example " The colouring of a regular map in 4 colours is equivalent to the colouring of its edges in 3 colours."

In the third lecture Prof. C. J. Keyser points out how Charles Sanders Pierce, (1839—1914) the son of Benjamin Pierce to whom we owe the definition of mathematics as "the Science which draws necessary conclusions", was a pioneer to whom the world owes many basic seed-thoughts which were later to become fruit giving trees. These seed thoughts are those connected with "pragmatism", "propositional function", the "logic of relations" and the infinities of cantor.

In the last lecture on "The Fourth Dimension and Relativity" Prof. Infeld gives in the form of a conversation the concept of the 4-dimensional space-time continuum as a static picture of the world of events.

A. N. RAO

Projective Geometry

By S. Suryanarayana Iyer, M.A. and A. Santiago, M.A. with a foreword by Rev. C. Racine, S.J., St. Joseph's Industrial School Press, Trichinopoly, 1942, pp. 374.

This is a laudable attempt on the part of the authors to produce a text-book suitable for the B. A. Pass and Honours classes of the South Indian Universities, covering roughly the same ground as Askwith's *Pure Geometry.* Some of the topics receive ampler treatment such as Newton's Theorem and plane perspective. A fallacy in the direct application of Newton's Theorem with one of the directions parallel to an asymptote was pointed out by the first of the two authors in *The Mathematics Student* vol. IV p. 82.

One welcome improvement over the usual treatment is the illustrative "Analytical treatment" of the results in pure geometry, though this might with advantage have been carried still further, for example in the treatment of pencils and ranges of conics. The appropriate coordinate system for projective geometry is homogeneous coordinates, whether for points on a line or on a plane, and there is no need for the restriction $x+y+z=1$ given on page 322. Several of the results given in the chapter on circular points at infinity which are very properly treated by homogeneous co-ordinates, will be true only if the coefficients of the conic are assumed to be real. The authors devote one chapter to "involution" and long afterwards, another to "homographic ranges and pencils" apparently bearing no relation to the former, separate analytical treatments being given in the two cases. The homographic transformation or the linear operator which is the connecting link receives but scant recognition, if any at all.

The Syllabus in Geometry followed in many Indian Universities is in urgent need of revision. While the characteristic properties of conics may be allowed to be derived from those of a circle by projection, enough material should be included to show that Projective Geometry, far from being the handmaid of Metrical Geometry is a Queen in her own fair and extensive

domain which includes the metrical as a province thereof. Such a treatment would use analytical methods very freely with homogeneous co-ordinates, lay proper emphasis on the linear transformation, projective co-ordinates and the group concept, and if written so as to be suitable for use as a text book, should prove a stimulus for a revision of our courses. In remarking that the book under review is not of the above type, we are only stating that the authors have shown worldly wisdom and economic insight by meeting an existing need instead of seeking to create one. Still, one wishes there were less of worldly wisdom in the world!

A. N. RAO

A Manual of Geometry

By N. K. Narasimhamurthy M. Sc., pp. 120 Price Rs. 1—4—0.
The Prabhakar Book Depot, Bangalore.

A Manual of Trigonometry

By N. K. Narasimhamurthy M. Sc., pp. 176+16. Price Rs. 2—0—0.
The Prabhakar Book Depot, Bangalore.

Both these books are intended for the students of the Intermediate classes of the Mysore University. The Geometry Manual deals with the geometrical representation of algebraic identities, medial section, ratio and proportion, similar figures and some propositions in elementary solid geometry. The Trigonometry manual covers much the same ground as Loney's Trigonometry, Part I and contains 16 pages of mathematical tables. The treatment in both the books is brief but clear, and there are explanatory notes to bring out points requiring special attention. The printing and get up are good, and the books should be useful to those for whom it is intended.

A. N. RAO

The Syracusans, who had been in former times so fond of the sciences, did not long retain the esteem and gratitude they owed to a man who had done so much honour to their city. Less than a hundred and forty years after, Archimedes was so perfectly forgotten by his citizens, notwithstanding the great services he had done them, that they denied his having been buried at syracuse. It is Cicero who informs us of this circumstance.

At the time that he was quaestor in sicily, his curiosity induced him to make a search after the tomb of Archimedes. The syracusans assured him that his search would be to no purpose, and that there was no such monument among them. Cicero pitied their ignorance * * * At length, after several fruitless attempts he pereived without the gate of the city, * * * a pillar almost entirely covered with thorns and brambles, through which he could discern the figure of a sphere and a cylinder.

From ROLLIN: *History of the Ancients.*

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Indian Mathematical Society :

M. Abdulla Butt, Esq., M. A., Lecturer, Muslim University, Aligarh.

C. P. S. Menon, Esq., M.A., M.SC., F.R.A.S., the Doon School, Dehra Dun.

Ram Dhar Misra, Esq., M.Sc., Ph.D. (Edin) Lecturer, Lucknow University.

Akhowry Vindhyachal Prasad, Esq., M.Sc., Lecturer, Rajendra College, Chapra (Bihar).

P. Thirumalachary, Esq., B.Sc. (Hons.) C. M. A's Office, Poona.

V. V. Kale, Esq., B.E., Asst. Engineer, P. W. D., Bombay.

A. S. Ranganathan, Esq., B.A., 12 A, Cleaveland Town, F Street, Bangalore Cantonment.

R. K. Rubugunday, Esq., B.A. (Cantab), 1/4 North Mada Street, Mylapore, Madras.

Ambikeshwar Sharma, Esq., Birla College, Pilani (Jaipur State).

C. B. Rathie, Esq., Asst. Prof. of Mathematics, Dungar College, Bikaner (Rajaputana).

F. J. Noronha B.A., Hons. (Lond), Lecturer, Central College, Bangalore.

K. N. P. Nair, Esq., Doon College, Dehra Dun.

V. Ramaswami B.A. Hons. (Madras), Ph.D. (Cantab), Reader, Andhra University, Guntur.

The Managing Committee of the Indian Mathematical Society has been reconstituted as follows:

President : Dr. F. W. Levi, Hardinge Professor of Mathematics, Calcutta University.

Secretaries : Prof. S. Mahadevan, Presidency College, Madras.
Dr. M. R. Siddiqi, Osmania University, Hyderabad.

Librarian : Prof. R. P. Shintre, Fergusson College, Poona.

Treasurer : Prof. L. N. Subramaniam, Christian College, Tambaram.

Other Members : Prof. D. D. Kosambi, Fergusson College, Poona.
" J. Maclean, Wilson College, Bombay.
" A. N. Singh, Lucknow University, Lucknow.
" A. Narasinga Rao, Annamalai University, Annamalainagar.
" Ram Behari, St. Stephen's College, Delhi.
" N. R. Sen, Calcutta University, Calcutta.
" S. S. Pillai, Calcutta University, Calcutta.

The authorities of the Annamalai University have invited the Indian Mathematical Society to hold its next conference at Annamalainagar about the end of December 1943, and the Society has accepted the invitation. It is also proposed to hold, in connection with the conference, a mathematical exhibition intended to illustrate the richness and variety of the subject and the wide range of its applicability to life situations. Suggestions regarding suitable items, as well as charts, models, instruments and other exhibits will be gladly received and exhibits on loan duly acknowledged and returned at the end of the conference. It is also proposed to have a "book section" for exhibiting books on mathematics. All correspondence relating to the conference, and all papers to be read at the session may kindly be sent (with 2 short abstracts of each paper) to Dr. A. Narasinga Rao, Annamalainagar P. O., South India. It is hoped that, in spite of the difficult conditions under which the conference and exhibition are held the enthusiasm and cooperation of the members will make the venture a great success.

The Society has decided to suspend the Life Composition concessions till the end of the war.

The Ramanujam Memorial Prize of the Madras University for 1941 has been awarded jointly to Dr. S. Minakshisundaram and Dr. T. Venkatarayudu the titles of the winning theses being repsectively "A problem in Fourier expansion" and "Vibrations of a symmetric point system."

The University of London has conferred the Degree of Ph. D. on S. M. Shah, Esq., Lecturer in Mathematics in the Muslim University, Aligarh. The title of the Thesis was "On the relations between the number of zeros and the maximum modulus of an Integral function."

Mr. Nizamuddin M. Sc. has been appointed Lecturer in Mathematics in the Muslim University, Aligarh.

Prof. A. R. Forsyth whose weighty treatises on Function Theory and on Differential Equations have played an important part in the education of generations of mathematicians died June 1942 at the age of 84. Other Mathematicians who died recently and with whose names the students of higher mathematics will be familiar include:

Dr. Vito Volterra of Rome in October 1940 aged 80 years

Prof. Alfred Pringsheim of Munich in 1941 at the age of 91 years

Dean W. C. Graustein of Harvard in January 1941

Dr. E. L. Ince, of Edinburgh in 1941 at the age of 49 years

Prof. I. Schur of Berlin died in 1941